Schemes (Har II 2, Shaf V 3.1–2)

Def:

- An <u>affine scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.
- A <u>scheme</u> is a locally ringed space $(X, \mathcal{O}_X)$ such that every point has an open neighborhood U s.t. $(U, \mathcal{O}_X|_U)$ is an affine scheme

  $\mathcal{O}_X$ is called the <u>structure sheaf</u> of $(X, \mathcal{O}_X)$ (which is usually just written as X).
- A <u>morphism</u> of schemes is just a morphism of locally ringed spaces.

Remark: In the last section, we showed that the functor $R \longmapsto (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is an (anti-) equivalence of categories. That is, if we compose it with $(\operatorname{Spec} R, \mathcal{O}) \longmapsto \mathcal{O}(\operatorname{Spec} R)$ we get the identity (on objects and morphisms). The "anti" means that it reverses arrows. <u>Key takeaway</u>: an affine scheme is "essentially the same" as a ring: it is completely determined by the ring of its global sections.

Ex: If k is a field, then $\operatorname{Spec} k$ = a point, and the structure sheaf is just $\mathcal{O} = k$.

However, set $R = k[x]/(x^2)$. Then $\operatorname{Spec} R$ = a point but has a different structure sheaf.

The quotient map $R \to k$ induces the maps

$$\begin{aligned} f : \operatorname{Spec} k &\to \operatorname{Spec} R \\ (0) &\longmapsto (x) \end{aligned} \quad \text{and} \quad f^{\#} : R \to k.$$

Define $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ along w/ its structure sheaf.

Ex: $\mathbb{A}^1_k = \{ (p(x)) \mid p(x) \text{ monic and irreducible} \} \cup \underset{\substack{\uparrow \\ \text{generic point}}}{\{(0)\}}$

Ex: If $k = \bar{k}$, then

$$\mathbb{A}^2_k = \underbrace{\{ (x-a, y-b) \mid a, b \in k \}}_{\text{closed points}} \cup \underbrace{\{ (f(x,y)) \mid f \text{ irreducible (up to scaling)} \}}_{\text{curves}} \cup \underbrace{\{(0)\}}_{\text{generic point}}$$

If $f \in k[x,y]$ irreducible, then

$$f(x,y) \in (x-a, y-b) \iff f(a,b) = 0$$

i.e. the closure of f contains the closed points corr. to points on the curve. (f) is called the generic point of the curve $f(x,y) = 0$.

Ex: $\mathbb{A}^2_{\mathbb{R}}$ has additional points that have less concrete geometric meaning. e.g. $P = (x^2+y^2)$ is prime, but $\overline{\{P\}} = \{P, (x,y)\}$.

$Q = (x^2+1, y)$ is maximal and thus closed.

Remark: If $a \in R$, note that $D(a) \subseteq \operatorname{Spec} R$ is itself an affine scheme: The ring map

$$R \longrightarrow R_a$$

induces a morphism of affine schemes

$$f : \operatorname{Spec}(R_a) \longrightarrow \operatorname{Spec} R$$

which is a homeomorphism onto its image,

$$D(a) = \operatorname{Spec} R \setminus V(a) = \{P \in \operatorname{Spec} R \mid a \notin P\}.$$

Thus, we just need that $f^\#$ is an isomorphism, which we can check on stalks. Take $Q \in \operatorname{Spec}(R_a)$.

$$f^\#_P : \underset{\substack{\| \\ Q \cap R}}{R_{f(Q)}} \longrightarrow (R_a)_Q$$

is an isomorphism since $a \notin Q \cap R$.

Thus, $\left(D(a), \mathcal{O}_{\operatorname{Spec} R}\big|_{D(a)}\right) \cong \left(\operatorname{Spec} R_a, \mathcal{O}_{\operatorname{Spec} R_a}\right)$ as loc. ringed spaces.

Glueing schemes

We can build more complicated schemes by glueing affine schemes together. More generally:

Let X_1, X_2 be schemes, and $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ open subsets. First note that U_1, U_2 are both schemes themselves:

For $p \in U_i$, let $V \ni p$ be an affine open in X. Then $V \cap U_i$ is open and so contains a distinguished open neighborhood of p, which is also affine, by the remark above.

Suppose $\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \longrightarrow (U_2, \mathcal{O}_{X_2}|_{U_2})$ is an isomorphism. Then we can <u>glue</u> X_1 and X_2 along U_1 and U_2 to obtain a scheme X as follows:

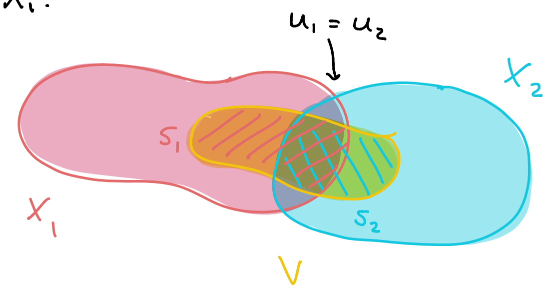
X has underlying topological space $X_1 \cup X_2 / \sim$, where $x \sim \varphi(x)$, given the quotient topology.

For $V \subseteq X$ open, define the structure sheaf as:

$$\mathcal{O}_X(V) = \left\{ (s_1, s_2) \;\middle|\; s_1 \in \mathcal{O}_{X_1}(X_1 \cap V),\ s_2 \in \mathcal{O}_{X_2}(X_2 \cap V) \text{ and } s_1|_{U_1 \cap V} = s_2|_{U_2 \cap V} \right\}$$

Note that this is the unique sheaf s.t. $\mathcal{O}_X(V) = \mathcal{O}_{X_i}(V)$

if $V \subseteq X_i$.



restricting s_2 to $U_2 \cap V$ and then taking image under isom. should give $s_1|_{U_1 \cap V}$.

Then $(X, \mathcal{O}_X)$ is a scheme.

Ex: Let $X_1 = \underset{\text{Spec } k[x]}{\overset{\shortparallel}{\mathbb{A}^1_k}} = X_2$ and $U_1 = \mathbb{A}^1_k - \{(x)\} = U_2$.

Let $\varphi : U_1 \longrightarrow U_2$ be the identity map, and let X be the scheme obtained by glueing X_1 and X_2 along U_1, U_2 via φ. This gives an affine line with a double origin:

What is $\mathcal{O}(X)$?

$$\mathcal{O}(X) = \{(a_1, a_2) \mid a_1, a_2 \in k[x] \text{ and } a_1|_{D(x)} = a_2|_{D(x)}\}$$

$\mathcal{O}(D(x)) = k[x]_x \supseteq k[x]$, so $\frac{a_1}{1} = \frac{a_2}{1}$ in $k[x]_x \iff a_1 = a_2$.

Thus $\mathcal{O}(X) = k[x]$.

However, $X \not\cong \operatorname{Spec} k[x]$ (do you see why?), so X is not an affine variety.

We'll come back to this example in a few weeks.